Linear Algebra I Summary

Vectors and Products

- Vectors starting at different points are equal if their normal and direction are
- **Parallelogram law**: adding two vectors corresponds to finding the diagonal of the parallelogram with those vectors as adjacent sides
- $c\vec{v} \implies c = 0$ or $\vec{v} = \vec{0}$
- **Standard basis vectors**: $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$
  - These form the *standard basis* for $\mathbb{R}^n$ (or $\mathbb{C}^n$, if we are considering complex numbers)
- **Dot product**: `vector` → `scalar`: sum of pairwise multiplication of corresponding vector elements
  - Distributivity: $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
  - Linearity: $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$
  - $\vec{v} \cdot \vec{v} \geq 0$, and is only 0 when $\vec{v} = \vec{0}$
- **Norm**: $\|v\|$ is defined as $\sqrt{\vec{v} \cdot \vec{v}}$, and corresponds to the *length* of the vector
  - Linearity: $\|c\vec{v}\| = |c|\|\vec{v}\|$
  - The **normalization** of a vector $\vec{v}$ is $\hat{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$, $\hat{v}$ is $\vec{v}$ with magnitude 1, i.e. $\hat{v} = 1$
- Angle between vectors in terms of dot product: $\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta}$
  - So, the magnitude of the dot product encodes the angle of the vectors
- **Orthogonal vectors**: perpendicular vectors, i.e. vectors where $\vec{u} \times \vec{v} = 0$
  - All vectors are orthogonal to $\vec{0}$
  - $\begin{bmatrix} a \\ b \end{bmatrix}$ is orthogonal to $\begin{bmatrix} b \\ -a \end{bmatrix}$, etc.
- **Projection** of $\vec{v}$ onto $\vec{w}$: $\boxed{\operatorname{proj}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \hat{w})\hat{w} = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w}}$
  - This finds the components of $\vec{v}$ that is parallel to $\vec{w}$
  - Perpendicular component of $\vec{v}$ to $\vec{w}$: $\operatorname{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \operatorname{proj}_{\vec{w}}(\vec{v})$
  - perp and proj are always orthogonal
- **Standard Inner Product**: $\langle \vec{v}, \vec{w} \rangle = v_1\overline{w_1} + v_2\overline{w_2} + \cdots + v_n\overline{w_n}$
  - *Generalization of the dot project that can be used for* $\mathbb{C}$
  - Linearity of first argument: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
- **Cross product**: for $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\vec{u} \times \vec{a} = \begin{bmatrix} u_2a_3 - u_3a_2 \\ -(u_1a_3 - u_3a_1) \\ u_3a_2 - u_2a_1 \end{bmatrix}$

- The product $\vec{u} \times \vec{a}$ will be orthogonal to both $\vec{u}$ and $\vec{a}$
- $\boxed{\vec{u} \times \vec{a} = \|\vec{u}\|\|\vec{a}\| \sin\theta}$
- Skew symmetry: $\vec{u} \times \vec{a} = -\vec{a} \times \vec{u}$
- The cross product is linear in both arguments

Linear Combinations

- **Linear combination**: $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$
- **Span**: set of all possible linear combinations: $\mathrm{Span}\{\vec{v_1}, \ldots, \vec{v_n}\}$
  - $\vec{0}$ is always in a span of vectors
  - The span of one vector is a line
  - Is $\begin{bmatrix}0\\1\\0\end{bmatrix}$ in $\mathrm{Span}\left\{\begin{bmatrix}2\\0\\5\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix}\right\}$? Solve $\begin{bmatrix}0\\1\\0\end{bmatrix} = a\begin{bmatrix}2\\0\\5\end{bmatrix} + b\begin{bmatrix}1\\0\\3\end{bmatrix}$ to find a system of linear equations
- Vector equation of line with slope $\dfrac{q}{p}$ through point $\begin{bmatrix}a\\b\end{bmatrix}$: $\boxed{\vec{\ell} = \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x_1\\y_1\end{bmatrix} + t\begin{bmatrix}p\\p\end{bmatrix}, t \in \mathbb{R}}$
  - This can generalize to $\mathbb{R}^n$
  - A line parallel to this has the same direction vector
- Line through $\vec{u}$ with direction $\vec{v}$ as a set of points (vectors): $\boxed{\mathcal{L} = \{\vec{u} + t\vec{v} : t \in \mathbb{R}\}}$
- Line through vectors $\vec{p}$ and $\vec{q}$: $\boxed{\vec{\ell} = \vec{u} + t(\vec{q} - \vec{u}), t \in \mathbb{R}}$ ($\vec{u}$ is a parameter)
- Plane through origin: $\boxed{P = \mathrm{span}\{\vec{v}, \vec{w}\} = \{s\vec{v} + t\vec{w} + \vec{a} : s, t \in \mathbb{R}\}}$
  - Without the constant offset $\vec{a}$, this is a plane through the origin
  - $\vec{v}$ and $\vec{u}$ are the direction vectors
- Plane through points $\vec{v}, \vec{q}, \vec{r}$: $\boxed{P = \{\vec{u} + s(\vec{q} - \vec{u}) + t(\vec{r} - \vec{u}) : s, t \in \mathbb{R}\}}$
- Finding a *scalar equation of a plane*: find normal vector to the plane using the cross product, equate $\begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix} = 0$, by the dot product definition, $n_1x + n_2y + n_3z = 0$
- Systems of equations arise from questions about spans
- If $\vec{v} \in \mathrm{Span}\{\vec{u}, \vec{w}\}$, then $\vec{v} = a\vec{u} + b\vec{w}$, meaning $v_1 = au_1 + bw_1$, $v_2 = au_2 + bw_2$, etc.
  - Asking this is like asking if $\vec{v} = a\vec{u} + b\vec{w}$ has any solutions
  - A solution set has 0, 1, or ∞ solutions ($0 \iff$ **inconsistent** system of equations)
  - The *equivalent system* has the same solution set
- EROs (**elementary row operations**) always produce equivalent systems
  - Options: switching rows, scaling a row by non-0, setting one row as a linear combination of others
  - $0v_1 + 0v_2 + \cdots = 0$ is a trivial row that doesn't add any information
  - **Row equivalency**: a matrix can be transformed into another only using EROs

- **REF (Row Echelon Form)**: 0-rows all at the end, leading entry appears to the right of term above
- **RREF (Reduced Row Echelon Form)**: In REF, all pivots are 1, pivots are the online nonzero entry in their column
    - RREF is unique for a matrix
    - If the system is inconsistent, then $[0 \dots 0|b]$ will appear
- Gauss-Jordan elimination: Matrix → REF row by row downward, REF → RREF row by row upward
- **Rank**: Matrix $A_{m \times n}$ has r pivots in RREF (and/or REF) $\iff \text{Rank}(A) = r$
    - $\text{Rank}(A) \leq \min\{m, n\}$
- System is consistent $\iff \text{Rank}(A) = \text{Rank}([A|\vec{b}])$
- Let $A \in M_{m \times n}(\mathbb{F})$ with $\text{Rank}(A) = r$
    1. If $[A|\vec{b}]$ is consistent, the solution set has $n - r$ parameters (degrees of freedom)
    2. $[A|\vec{b}]$ is consistent for every $\vec{b} \iff r = m$
- The **nullity** of $A_{m \times n} = n - \text{Rank}(A)$ is the number of parameters
- Homogeneous system: $\vec{b} = [0 \dots 0]^T$
    - Always has a solution → Always consistent for any A
- The **null space** of A, $\text{Null}(A)$, is the solution to $[A|\vec{0}]$
    - This can always be written as the span of vectors
- The solution sets of $[A|\vec{b}]$ and $[A|\vec{0}]$ will only differ by a constant offset vector
- Reminder for $\mathbb{C}$: $\dfrac{1}{z} = \dfrac{z}{z \times \overline{z}} = \dfrac{a - bi}{a^2 + b^2}$
- $A\vec{x} = \vec{b}$ encodes the whole system of equations $x_1 \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \end{bmatrix} + x_2 \begin{bmatrix} a_{2,1} \\ a_{2,2} \\ \vdots \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{n,1} \\ a_{n,2} \\ \vdots \end{bmatrix}$
    - Linear combination of columns of A by $b\vec{x}$ equals $\vec{b}$
    - $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$, $A(c\vec{x}) = c(A\vec{x})$
- If $A\vec{x} = \vec{e_i}$ is consistent for every standard basis vector in $\mathbb{F}^m$, then $\text{Rank}(A) = m$
- Let the solution of $A\vec{x} = \vec{0}$ be S. If $\vec{x}, \vec{y} \in S$ and $c \in \mathbb{F}$, then $\vec{x} + \vec{y} \in S$, $c\vec{x} \in S$, and $c_1\vec{x} + c_2\vec{y} \in S$, etc
- Let $A\vec{x} = \vec{b}$, where $\vec{b} \neq \vec{0}$ be consistent, with solution $\tilde{S}$. Let $A\vec{x} = \vec{0}$ be the **associated homogeneous system** with solution set S. Then, if $x_p \in \tilde{S}$, $\tilde{S} = \{\vec{x_p} + \vec{x} : \vec{x} \in S\}$
    - If $A\vec{x} = \vec{b}$ is consistent, it has the same number of solutions/parameters as $A\vec{x} = \vec{0}$
- Let $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ be consistent with $\vec{b} \neq \vec{c} \neq \vec{0} \neq \vec{b}$. If their solution sets are $\tilde{S_b}$ and $\tilde{S_c}$ with particular solutions $\vec{x_b}$ and $\vec{x_c}$, then $\boxed{\tilde{S_c} = \{(\vec{x_c} - \vec{x_b}) + \vec{z} : \vec{z} \in \tilde{S_b}\}}$
    - I.e. the second solution set is just an *offset* of the first one
- **Column Space** of A, $\text{Col}(A)$: the span of the columns of A
    - $\boxed{A\vec{x} = \vec{b} \text{ is consistent} \iff \vec{b} \in \text{Col}(A)}$; this is use to show *consistency*
- **Transpose** of a matrix A^T: for $i, j \in \mathbb{N}$, switch A_{ij} with A_{ji}

- **Row Space** of A, $\text{Row}(A)$: the span of the rows of A, if treated as column vectors
  - $\text{Row}(A) = \text{Col}(A^T)$
- Performing EROs does *not* affect the row space, since the mirror the operations used to populate the span
- **Matrix Equality**: matrices are the same size and every corresponding element is equal
- **Column extraction**: A_{e_i} is the ith column of A
  - Matrix equality test: $\boxed{A = B \iff A\vec{x} = B\vec{x} \text{ for all } x \in \mathbb{F}^n}$
- Matrix multiplication example:
$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right] = \begin{bmatrix} 3 & -5 \\ 7 & -11 \\ 6 & -4 \end{bmatrix}$$
  - $\#$ of *cols* in the first matrix $= \#$ of *rows* in the second matrix
  - Matrix multiplication is *non-commutative*
  - Every column of $C = AB$ is a member of $\text{Col}(A)$
  - (i, j)th entry of C is the dot product of column i of A and column j of B
- **Distributivity**: $(A + B)C = AC + BC$ (*right-handed*), $A(C + D) = AC + AD$ (*left-handed*)
  - Come in different-handed versions since matrix multiplication *isn't commutative*
- **Associativity**: $ACE = A(CE) = (AC)E$
- For $s \in \mathbb{F}$
  - $s(A + B) = sA + sB$
  - $s(AB) = (sA)B = A(sB) = sAB$
- The *cancellation law* only holds if A is invertible, i.e. $AB = AC \text{ and } A \neq \mathcal{O} \not\Longrightarrow B = C$ unless A is invertible
  - Similarly, $AB \neq \mathcal{O} \not\Longrightarrow A = 0 \text{ or } B = 0$ unless A is invertible
- Transposes can be *added* and *scaled*, then converted without difference, i.e. $\boxed{(AB)^T = B^T A^T}$
- **Identity matrix**: $\boxed{I_m A = A \text{ and } AI_n = A}$; the same holds for vectors
- **Elementary matrix**: result of one ERO performed on I
  - These are used to encode EROs and carry them out by multiplication
  - This matrix is found my performing the same ERO on I
  - These can be *chained together*: $D = E_k \times E_{k-1} \times \cdots \times E_2 \times E_1 \times A$
- **Invertibility**: A matrix is **invertible** if it is $n \times n$, and there exists $E \in M_{n \times n}$ such that $AB = CA = I_n$
  - In this case, we must have $B = C$
  - If B exists, C must also exist, and vice-versa
  - We denote this as the **inverse** A^{-1}, where $AA^{-1} = A^{-1}A = I_n$
- A is invertible $\iff \text{Rank}(A) = n \iff \text{RREF}(A) = I_n$
- Inverse matrices can be found by solving the augmented matrix $[A|I_n]$ into RREF form. If A becomes I_n, then I_n has become the inverse. Otherwise, A is not invertible.

Chapter 5 - Linear Transformations

- Function determined by matrix A is $T_A : \mathbb{F}^n \to \mathbb{F}^m$, where $\boxed{T_A : \vec{x} \mapsto A\vec{x}}$
- T is linear $\iff T(c\vec{x} + \vec{y}) = cT(\vec{x}) + T(\vec{y})$
- $\vec{0}$ **maps to** $\vec{0}$: We always have $T(\vec{0}) = \vec{0}$
- We prove non-linearity by counterexample
- Functions that don't "look" linear often aren't, e.g. $\vec{x}^2$, $\vec{x}\vec{y}$, $\sqrt{\vec{y}}$, etc.)
- We have $\text{Range}(T_A) = \text{Col}(A)$
  - **Range**: set of values that could possibly be achieved by a linear transformation of A
- $T : \mathbb{F}^n \to \mathbb{F}^m$ is **onto/surjective** iff $\text{Range}(T) = \mathbb{F}^m$, which happens iff $\text{Rank}(A) = m$
- $\ker(T_A) = \text{Null}(A)$, i.e. the solution set to $A\vec{x} = \vec{0}$, i.e. the set of inputs to T where the output is 0
- **One-to-one/surjective** transformation: $T(\vec{x}) = T(\vec{y}) \implies \vec{x} = \vec{y}$
  - Distinct pairs of element in $\mathbb{F}^n$ and $\mathbb{F}^m$ are mapped together
  - This occurs iff $\text{Rank}(A) = n$

Invertibility Criteria

A is invertible
T_A is invertible
T_A is one-to-one
T_A is onto
$\text{Null}(A) = \{\vec{0}\}$ (only a trivial solution to $A\vec{x} = \vec{0}$)
$\text{Col}(A) = \mathbb{F}^n$ ($A\vec{x} = \vec{b}$ is always consistent)
$\text{Nullity}(A) = 0$
$\text{Rank}(A) = n$
$\text{RREF}(A) = I_n$

- Every linear transformation has a matrix $[T]_\varepsilon(\vec{x})$ where $[T]_\varepsilon = [T(\vec{e_1}) \quad T(\vec{e_2}) \quad \ldots \quad T(\vec{e_n})]$
  - T is *onto* $\iff \text{Rank}([T]_\varepsilon) = m$
  - T is *one-to-one* $\iff \text{Rank}([T]_\varepsilon) = n$
- Projection, reflection, and rotation are linear transformations
  - Counter-clockwise rotation by θ: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- *Composition* of linear functions $(T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x})) = [T_2]_\varepsilon[T_1]_\varepsilon(\vec{x})$
  - This is guaranteed to be a linear function

Chapter 6 - Determinants

- The **determinant** gives some information about a matrix; expresses itself as a *scaling factor*

    - $\det [a] = a$, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
    - Larger matrices: expand along a row or column, sum together all the matrices formed by removing the current row can column times the current value, with the sign following a checkerboard pattern in the matrix
- Matrix has a zero row → $\det A = 0$
- **Upper-triangular** matrices have the determinant equal to the diagonal entries
- EROs affect the determinant:
    - Row swap: $\det A = -\det A$
    - Row scale by m: $\det A = m \det A$
    - Row addition: *no change* to the determinant
- $\boxed{\det(AB) = \det(A)\det(B)}$
    - This can be chained together arbitrarily
- $\boxed{A \text{ is invertible} \iff \det A \neq 0}$
- $\boxed{\det(A^{-1}) = \frac{1}{\det A}}$, assuming that A is invertible
- **Cofactor matrix**: determinants of each row/col removed submatrix with alternating signs
- **Adjugate matrix**: transpose of the *cofactor matrix*
- $A^{-1} = \frac{1}{\det A} \times \operatorname{adj}(A)$
- **Cramer's Rule**: Let $A\vec{x} = \vec{b}$. If we replace column j of A with $\vec{b}$ to get A_b, then the solution to $A\vec{x} = \vec{b}$ is $\vec{x}_j = \frac{\det B_j}{\det A}$
    - We can use $j = 1 \dots n$ to find the whole solution vector $\vec{x}$
- Determinant indicates how much *multiplying by a matrix scales space*
    - Negative determinant → spaces was "flipped"

Chapter 7 - Eigenvectors and Eigenvalues

- Sometimes, a transformation just *scales* a vector instead of changing its direction, i.e. $A\vec{x} = \lambda\vec{x}$, where $\lambda \in \mathbb{F}$
    - Such a vector $\vec{x}$ is an **eigenvector**, and its scaling factor λ is its **eigenvalue**
- **Eigenvalue equation**: $A\vec{x} = \lambda\vec{x} \iff (A - \lambda I)\vec{x} = \vec{0}$
- **Characteristic polynomial** $C_A = \det(A - \lambda I) = 0$ solves for eigenvalues
    - The *highest term* c_n is $(-1)^n$
    - $c_{n-1} = (-1)^{n-1} \times \operatorname{trace}(A)$
    - The *constant term* c_0 is $\det(A)$
    - In $\mathbb{C}$, we have $\sum_{i=1}^{n} \lambda_i = \operatorname{trace}(A)$ and $\prod_{i=1}^{n} \lambda_i = \det(A)$ (these both follow)
- We can find an eigenvector by plugging an eigenvalue into $(A - \lambda I)\vec{x} = \vec{0}$
- Any scalar multiple of $\vec{x}$ is trivially also an eigenvalue

- **Eigenspace** of A: $E_\lambda(A) = \text{Null}(A - \lambda I) = \text{ solution set of } (A - \lambda I)\vec{x} = \vec{0}$
- Expressing A as PDP^{-1} (where D is a diagonal matrix) makes it easier to compute A^k
- A **is similar to** $B \to PBP^{-1} = A$ for some P
    - If A and B are similar, the have the same *eigenvalues*, *characteristic polynomial*, and *determinant*
- **Diagonalizable**: $A = PDP^{-1}$ (where D is a diagonal matrix)
    - A will have n eigenvalues, which will be the diagonal entries of D
    - A has n *distinct* eigenvalues $\iff$ A is diagonalizable, and $P = [\vec{v_1} \dots \vec{v_n}]$ consists of the *eigenvectors of* A

Chapter 8 - Subspaces and Bases

- Subset $V \subseteq \mathbb{F}^n$ is a **subspace** if it is closed under addition and multiplication, and $\vec{0} \in V$
    - Essentially, the a subspace is the span of any subset of $\mathbb{F}^n$
    - $\{\vec{0}\}$, $\text{Span}(V)$, $\text{Null}(A)$, $\text{Col}(A)/\text{Range}(T)$ for any A are subspaces
    - Eigenspaces of matrices are subspaces
- $\boxed{V \subseteq \mathbb{F}^n \text{ is a subspace } \iff V \neq \emptyset \text{ and } \forall \vec{x}, \vec{y} \in V, c \in \mathbb{F}, \vec{x} + c\vec{y} \in V}$
- $\vec{v_1} \dots \vec{v_k} \in \mathbb{F}^n$ are **linearly dependent** if we have some $c_1\vec{v_1} + \dots + c_k\vec{v_k} = 0$, where not all $c_1 \dots c_k$ are 0
    - At least one vector is a linear combination of others
    - If $c_1 = \dots = c_k = 0$ is the only solution, the set is **linearly independent**
- $B = \{\vec{v_1} \dots \vec{v_k}\} \subset V$ is a **basis** for subspace V if B is *linearly independent* and $\text{Span}(B) = V$
    - Everything in B can be constructed from B's vectors
- Let A be the $n \times k$ matrix $[\vec{v_1} \dots \vec{v_k}]$
- $\{\vec{v_1} \dots \vec{v_k}\}$ is *linearly independent* $\iff$ $\text{Rank}(A) = k$ (i.e. it has no pivots)
    - The set of vectors that correspond to $\text{RREF}(A)$ pivots are a linearly independent set with span $\text{Span}\,\{\vec{a_1} \dots \vec{a_k}\}$
    - Adding a non-pivot vector makes the set linearly dependent
- A set of more than n vectors in $\mathbb{F}^n$ *must* be linearly dependent
- Every subspace has a spanning set $\text{Span}\,\{\vec{v_1} \dots \vec{v_k}\} = V$
    - $S \subseteq V \implies \text{Span}(S) \subseteq V$
    - $\text{Span}(S) = \mathbb{F}^n \iff \text{Rank}([S]) = n$
- **Every subspace has a basis** $\to$ Any basis for $\mathbb{F}^n$ must have n vectors
- $B = \{\vec{v_1} \dots \vec{v_k}\}$ spans $\mathbb{F}^n$ $\iff$ B is *linearly independent*
- Set of Pivot columns of A (not necessarily $RREF(A)$) is a basis for $\text{Col}(A)$
- If $\text{Null}(A) = \{t_1\vec{x}_1 + \dots + t_k\vec{x}_k : t_{1 \to k} \in \mathbb{F}\}$, then $\{\vec{x}_1 \dots \vec{x}_k\}$ is a basis for $\text{Null}(A)$
- All bases for a set have the same number of vectors (i.e. **dimension**)
- For $A \in M_{m \times n}\mathbb{F}, n = \text{Rank}(A) + \text{Nullity}(A) = \dim(\text{Col}(A)) + \dim(\text{Null}(A))$

- Let V be a subspace of $\mathbb{F}^n$ with basis $B = \{\vec{v}_1 \dots \vec{v}_k\}$. There exist unique $c_1 \dots c_k \in \mathbb{F}$ such that $\vec{w} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$ for an $\vec{w} \in V$
  - For ε, these are just the components of the vector
  - **Coordinate vector of $\vec{w}$ with respect to B**: $[\vec{v}]_B = [c_1 \dots c_n]^T$
  - Taking coordinates is a *linear transformation*
- **Change of basis from B to C**: ${}_C[I]_B = \left[[\vec{v_1}]_B \dots [\vec{v_k}]_B\right]$
  - $[\vec{x}]_C = {}_C[I]_B[\vec{x}]_B$, $[\vec{x}]_B = {}_B[I]_C[\vec{x}]_C$ (basis of $\vec{x}$ changes)
  - ${}_C[I]_B$ is invertible and its inverse is ${}_B[I]_C$
- $[T]_B = T[[T(\vec{v_1})]_B, \dots, [T(\vec{v_k})]_B]$, $[T(\vec{v})]_B = [T]_B[\vec{v}]_B$
- $[T]_C = {}_C[I]_B[T]_B\,{}_B[I]_C = ({}_B[I]_C)^{-1}[T]_B{}_B[I]_C$
  - $[T]_B$ and $[T]_C$ are *similar* over $\mathbb{F}$
- Finding the standard matrix: $[T]_\varepsilon = {}_\varepsilon[I]_B[T]_B{}_B[I]_\varepsilon$
- $(\lambda, \vec{x})$ is an eigenpair of $T \iff (\lambda, [\vec{x}]_B)$ is an eigenpair of $[T]_B$
- T is diagonalizable over $\mathbb{F} \iff$ there exists an ordered basis consisting of the eigenvectors of T
  - P will be the matrix consisting of these eigenvectors (in the same order as the basis)
  - D will be $\mathrm{diag}(\lambda_1 \dots \lambda_k)$, i.e. the eigenvalues as diagonal entries
- T is diagonalizable $\iff [T]_B$ is diagonalizable
- Eigenvectors corresponding to unique eigenvalues are linearly independent $\implies P$ is invertible, as expected
- **Algebraic multiplicity** of λ_i is the power of $(\lambda - \lambda_i)$ in $C_A(\lambda)$
- **Geometric multiplicity** of λ_i is the dimension of the eigenspace of λ_i
  - Turns out to be $\mathrm{Nullity}(A - \lambda_i I)$
- We have $1 \leq g_{\lambda_i} \leq a_{\lambda_i}$
- The *union of bases of distinct eigenspaces* is linearly independent
- Diagonalizability test: diagonalizable $\iff C_A(\lambda)$ does not have an irreducible term and $a_{\lambda_i} = g_{\lambda_i}$ for all i
- Let $B = P^{-1}AP$, so A and B are similar
  - Then $B^k = P^{-1}AP$
  - If B is diagonal and P diagonalizes A, then $B^k = \mathrm{diag}(\lambda_1^k \dots \lambda_n^k)$

Unit 10: Vector Spaces

- **Addition** $\oplus$: combines two elements in a vector spaces into a vector, i.e. $\text{vector} \oplus \text{vector} \to \text{vector}$
- **Scalar multiplication** $\odot$: $\text{scalar} \odot \text{vector} \to \text{vector}$

Vector Space Axioms

$\mathbb{V}$ is a vector space over the *field* $\mathbb{F}$ if, under operations $\oplus$ and $\odot$, we have

- `C1` closure under addition $\oplus$
- `C2` closure under scalar multiplication $\odot$
- `V1` commutative addition
- `V2` associative addition
- `V3` additive identity
- `V4` additive inverse
- `V5` vector addition $\oplus$ distributive law
- `V6` scalar addition distributive law
- `V7` associative scalar multiplication
- `V8` multiplicative identity

- $\mathbb{F}^n$, $A_{m\times n}(\mathbb{F})$, $T : \mathbb{F}^m \to \mathbb{F}^n$, polynomials with degree $\leq n$ are all vector spaces
- The **zero space**: $\mathbb{V} = \{\vec{0}\}$
- The **zero vector** and **additive inverse** are unique in a vector space
- $0 \odot \vec{x} = \vec{0}$ and $a \odot \vec{0} = \vec{0}$ for all $\vec{x} \in \mathbb{V}, a \in \mathbb{F}$
- $-\vec{x} = (-1) \odot \vec{x}$
- Cancellation law: $a \odot \vec{x} = \vec{0} \implies a = 0 \text{ or } \vec{x} = \vec{0}$
- Linear combination and span apply to vector spaces
- $\mathbb{U} \subseteq \mathbb{V}$ is a subspace of $\mathbb{V}$ if $\mathbb{U}$ is a non-empty (i.e. contains a $\vec{0}$) and is closed under *addition* and *scalar multiplication*
- Let $\mathbb{V}$ be a vector spaces over $\mathbb{F}$ with $W = \{\vec{v}_1 \ldots \vec{v}_k\} \subseteq \mathbb{V}$
  - $\mathrm{Span}(W)$ is a subspace of $\mathbb{V}$
  - If $\mathbb{U}$ is a subspace where $W \subseteq \mathbb{U}$, then $\mathrm{Span}(W) \subseteq \mathbb{U}$
- B is a basis for $\mathbb{V}$ if B is linearly independent and $\mathrm{Span}(B) = \mathbb{V}$
- Unique representation theory holds for vector spaces as well; so do coordinate vectors, change of basis matrices, etc.